ToFu geometric tools
Intersection of a LOS with a cone

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Chapter 1

Definitions

1.1 Geometry definition in ToFu

The definition of a fusion device in ToFu is done by defining the edge of a poloidal plane as a set of segments in a 2D plane. The 3D volume is obtained by an extrusion for cylinders or a revolution for tori. We consider an orthonormal direct cylindrical coordinate system $(O, \underline{e}_R, \underline{e}_\theta, \underline{e}_Z)$ associated to the orthonormal direct cartesian coordinate system $(O, \underline{e}_X, \underline{e}_Y, \underline{e}_Z)$. We suppose that all poloidal planes live in (R, Z) and can be obtained after a revolution around the Z axis of the user-defined poloidal plane at $\theta = 0$, $\mathcal{P}_0$. Thus, the torus is axisymmetric around the (O, Z) axis (see Figure ??).

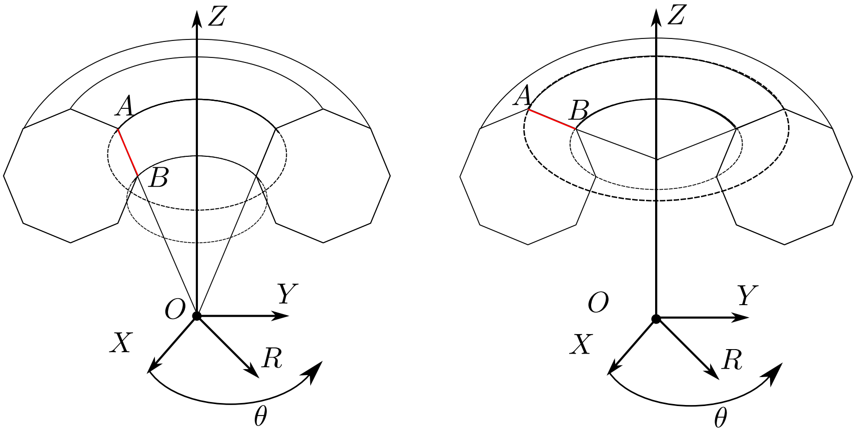

Figure 1.1: Two examples of a circular torus approximated by a revolved octagon. For each segment $\overline{AB}$ of the octagon there is a cone with origin on the (O, Z) axis.

1.2 Notations

In order to simplify the computations, let A and B be the end points of a segment $\mathcal{S}_i$ such that $A \neq B$ and $\mathcal{P}_0 = \cup_{i=1}^{n} \mathcal{S}_i = \cup_{i=1}^{n} \overline{\mathrm{A}_i \mathrm{B}_i}$ with n the number of segments given by the user defining the plane $\mathcal{P}_0$. We define a right circular cone $\mathcal{C}$ of origin $P = (\mathrm{A}, \mathrm{B}) \cap (\mathrm{O}, \mathrm{Z})$ of generatrix (A, B) and of axis (O, Z) (see Figure ??). Thus we can define the edge of the torus as the union of the edges of the frustums $\mathcal{F}_i$ defined by truncating the cones $\mathcal{C}_i$ to the segment $\overline{AB}_i$.

Then, any point M with coordinates (X, Y, Z) or (R, θ, Z) belongs to the frustum $\mathcal{F}$

if and only if

$$\exists q \in [0;1] / \left\{ \begin{array}{l} R - R_A = q(R_B - R_A) \\ Z - Z_A = q(Z_B - Z_A) \end{array} \right.$$

Now let us consider a LOS L (i.e.: a half-infinite line) defined by a point D and a normalized directing vector u, of respective coordinates (X_D, Y_D, Z_D) or (R_D, θ_D, Z_D) and (u_X, u_Y, u_Z). Then, point M belongs to L if and only if:

$$\exists k \in [0; \infty[/ \underline{DM} = k\underline{u}$$

Chapter 2

Derivation

Let us now consider all intersections between the edge of a frustum $\mathcal{F}$ and a semi-line L.

$$\exists(q,k) \in [0;1] \times [0;\infty[/ \quad \begin{cases} R - R_A = q(R_B - R_A) \\ Z - Z_A = q(Z_B - Z_A) \\ X - X_D = ku_X \\ Y - Y_D = ku_Y \\ Z - Z_D = ku_Z \end{cases} \tag{2.0.1}$$

Which yields (by combining to keep only unknowns q and k):

$$\begin{gathered} q(Z_B - Z_A) = Z_D - Z_A + ku_Z \\ q^2(R_B - R_A)^2 + 2qR_A(R_B - R_A) = \left(k\underline{u}_{//} + \underline{D}_{//}\right)^2 - R_A^2 \end{gathered} \tag{2.0.2}$$

Where we have introduced $R_D = \sqrt{X_D^2 + Y_D^2}$, $\underline{u}_{//} = u_X \underline{e}_X + u_Y \underline{e}_Y$ and $\underline{D}_{//} = X_D \underline{e}_X + Y_D \underline{e}_Y$. We can then derive a decision tree.

Given that the parallelization will take place on the LOS (i.e.: not on the cones which are parts of the vacuum vessel), we will discriminate case based prioritarily on the components of $\underline{u}$ and D. We will detail only the cases which have solutions, in order to make it as clear as possible for implementation of an efficient algorithm. We will also only consider non-tangential solution, as we are looking for entry/exit points.

2.1 Horizontal LOS: $u_Z = 0$

Let us consider an horizontal LOS, such that $u_Z = 0$, then (**??**) becomes

$$\exists(q,k) \in [0;1] \times [0;\infty[/ \quad \begin{cases} R - R_A = q(R_B - R_A) \\ Z_D - Z_A = q(Z_B - Z_A) \\ X - X_D = ku_X \\ Y - Y_D = ku_Y \\ Z = Z_D \end{cases}$$

From here we can differentiate two cases regarding the frustum $\mathcal{F}$.

2.1.1 Plane Frustum: $Z_B = Z_A$

Let us consider first the case where $Z_B = Z_A$, when the frustum becomes an annulus on the (X, Y) plane, then we will have two different cases.

- $Z_D \neq Z_A \Rightarrow$ the cone and the LOS stand in different parallel planes $\Rightarrow$ no solution.
- $Z_D = Z_A \Rightarrow$ the cone stands in the same plane as the LOS (see **??**) $\Rightarrow$ infinity of solutions, we consider no solutions as this is a limit case with no clearly identified intersection.

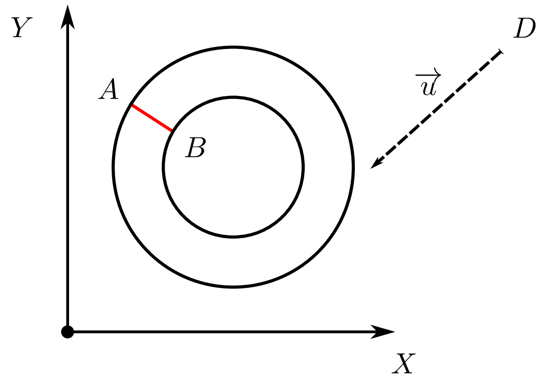

Figure 2.1: Plane frustum and horizontal Line of Sight on the same Z-plane.

Hence, the only derivable solutions suppose that $Z_B \neq Z_A$.

2.1.2 Non-horizontal cone: $Z_B \neq Z_A$

Then $q = \dfrac{Z_D - Z_A}{Z_B - Z_A}$. There are acceptable solution only if $q \in [0; 1]$. By introducing

$$C = q^2(R_B - R_A)^2 + 2qR_A(R_B - R_A) + R_A^2,$$

we have

$$\left(k\underline{u}_{//} + \underline{D}_{//}\right)^2 - C = 0 \Leftrightarrow k^2\underline{u}_{//}^2 + 2k\underline{u}_{//} \cdot \underline{D}_{//} + \underline{D}_{//}^2 - C = 0$$

Then introducing $\Delta = 4\left(\underline{u}_{//} \cdot \underline{D}_{//}\right)^2 - 4\underline{u}_{//}^2\left(\underline{D}_{//}^2 - C\right) = 4\delta$, there are non-tangential solutions only if $\left(\underline{u}_{//} \cdot \underline{D}_{//}\right)^2 > \underline{u}_{//}^2\left(\underline{D}_{//}^2 - C\right)$. It is necessary to compute the solutions k because we need to check if $k >= 0$.

$$k_{1,2} = \frac{-\underline{u}_{//} \cdot \underline{D}_{//} \pm \sqrt{\delta}}{\underline{u}_{//}^2}$$

Hence, we have solutions if:

$$\begin{cases} u_Z = 0 \\ Z_B \neq Z_A \\ \dfrac{Z_D - Z_A}{Z_B - Z_A} \in [0; 1] \\ k_{1,2} = \dfrac{-\underline{u}_{//} \cdot \underline{D}_{//} \pm \sqrt{\delta}}{\underline{u}_{//}^2} \geq 0 \end{cases}$$

2.2 Non-horizontal LOS: $u_Z \neq 0$

Then $k = q\dfrac{Z_B - Z_A}{u_Z} - \dfrac{Z_D - Z_A}{u_Z}$, which means:

$$
\begin{aligned}
q^2 \quad & (R_B - R_A)^2 + 2qR_A(R_B - R_A) + R_A^2 \\
&= \left(\left(q\frac{Z_B - Z_A}{u_Z} - \frac{Z_D - Z_A}{u_Z}\right)\underline{u}_{//} + \underline{D}_{//}\right)^2 \\
&= \left(q\frac{Z_B - Z_A}{u_Z} - \frac{Z_D - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2 + 2\left(q\frac{Z_B - Z_A}{u_Z} - \frac{Z_D - Z_A}{u_Z}\right)\underline{u}_{//} \cdot \underline{D}_{//} + \underline{D}_{//}^2 \\
&= q^2\left(\frac{Z_B - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2 - 2q\frac{Z_B - Z_A}{u_Z}\frac{Z_D - Z_A}{u_Z}\underline{u}_{//}^2 \\
&+ \left(\frac{Z_D - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2 + 2q\frac{Z_B - Z_A}{u_Z}\underline{u}_{//} \cdot \underline{D}_{//} - 2\frac{Z_D - Z_A}{u_Z}\underline{u}_{//} \cdot \underline{D}_{//} + \underline{D}_{//}^2
\end{aligned}
$$

Hence:

$$
\begin{aligned}
0 \quad = \quad & q^2\left((R_B - R_A)^2 - \left(\tfrac{Z_B - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2\right) \\
+ \quad & 2q\left(R_A(R_B - R_A) + \tfrac{Z_B - Z_A}{u_Z}\tfrac{Z_D - Z_A}{u_Z}\underline{u}_{//}^2 - \tfrac{Z_B - Z_A}{u_Z}\underline{u}_{//} \cdot \underline{D}_{//}\right) \\
- \quad & \left(\tfrac{Z_D - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2 + 2\tfrac{Z_D - Z_A}{u_Z}\underline{u}_{//} \cdot \underline{D}_{//} - \underline{D}_{//}^2 + R_A^2
\end{aligned}
$$

We can then introduce:

$$
\begin{cases}
A = (R_B - R_A)^2 - \left(\frac{Z_B - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2 \\
B = R_A(R_B - R_A) + \frac{Z_B - Z_A}{u_Z}\frac{Z_D - Z_A}{u_Z}\underline{u}_{//}^2 - \frac{Z_B - Z_A}{u_Z}\underline{u}_{//} \cdot \underline{D}_{//} \\
C = -\left(\frac{Z_D - Z_A}{u_Z}\right)^2 \underline{u}_{//}^2 + 2\frac{Z_D - Z_A}{u_Z}\underline{u}_{//} \cdot \underline{D}_{//} - \underline{D}_{//}^2 + R_A^2
\end{cases}
$$

Because of the shape of potential solutions, we have to discriminate the case $A = 0$.

2.2.1 $A = 0$: LOS parallel to one of the cone generatrices

Then, because of the shape of the potential solution, we have to discriminate the case $B = 0$. But in this case we have $C = 0$.

- if $C = 0 \Rightarrow$ no condition on q and k, the LOS is included in the cone $\Rightarrow$ we consider no solution
- if $C \neq 0 \Rightarrow$ Impossible, no solution

Only the case $B \neq 0$ is thus relevant.

$B \neq 0$: LOS not included in the cone

Then, there is either one or no solution:

$$
\begin{cases}
q = -\frac{C}{2B} & \in [0, 1] \\
k = q\frac{Z_B - Z_A}{u_Z} - \frac{Z_D - Z_A}{u_Z} & \geq 0
\end{cases}
$$

2.2.2 $A \neq 0$: LOS not parallel to a cone generatrix

Then, we only consider cases with two distinct solutions (i.e.: no tangential case):

$$\begin{cases} B^2 > AC & \\ q = \frac{-B \pm \sqrt{B^2 - AC}}{A} & \in [0, 1] \\ k = q\frac{Z_B - Z_A}{u_Z} - \frac{Z_D - Z_A}{u_Z} & \geq 0 \end{cases}$$

Appendix A

Acceleration radiation from a unique point-like charge

A.1 Retarded time and potential

A.1.1 Retarded time

Deriving the retarded time

Hence $\frac{dR(t_r)}{c} + dt_r = dt$

A.1.2 Retarded potentials

Deriving the potential propagation equations